GENERALIZED FRACTIONAL INTEGRAL OPERATORS AND THEIR MODIFIED VERSIONS

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ABSTRACT. Associated to a function \( \rho : (0, \infty) \to (0, \infty) \), let \( T_\rho \) be the operator defined on a suitable function space by

\[
T_\rho f(x) := \int_{\mathbb{R}^n} \frac{\rho(|x - y|)}{|x - y|^n} f(y) \, dy,
\]
and \( \tilde{T}_\rho \) be the modified version of \( T_\rho \) given by

\[
\tilde{T}_\rho f(x) := \int_{\mathbb{R}^n} \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) f(y) \, dy.
\]

For \( \rho(t) = t^\alpha, \ 0 < \alpha < n \), the operator \( T_\rho \) is nothing but the fractional integral operator or the Riesz potential, which is known to be bounded from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) provided that \( 1/p - 1/q = \alpha/n \).

Next, for \( 1 \leq p < \infty \) and a function \( \phi : (0, \infty) \to (0, \infty) \), we define the generalized Morrey space \( \mathcal{M}_\phi^p = \mathcal{M}_\phi^p(\mathbb{R}^n) \) by

\[
\mathcal{M}_\phi^p := \left\{ f \in L^p_{\text{loc}} : \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p \, dy \right)^{1/p} < \infty \right\}
\]
and the generalized Campanato space \( \mathcal{L}_\phi^p = \mathcal{L}_\phi^p(\mathbb{R}^n) \) by

\[
\mathcal{L}_\phi^p := \left\{ f \in L^p_{\text{loc}} : \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p \, dy \right)^{1/p} < \infty \right\},
\]
where the supremum is taken over all open balls \( B = B(a, r) \) in \( \mathbb{R}^n \), \( |B| \) denotes the Lebesgue measure of \( B \), \( \phi(B) = \phi(r) \), and \( f_B \) is the average of \( f \) over \( B \).

In this talk, we discuss the boundedness of \( T_\rho \) and \( \tilde{T}_\rho \) on generalized Morrey spaces and on generalized Campanato spaces, respectively. Under some conditions on \( \rho \), \( \phi \), and \( \psi \), we prove that \( T_\rho \) is bounded from \( \mathcal{M}_\phi^p \) to \( \mathcal{M}_\psi^q \), while \( \tilde{T}_\rho \) is bounded from \( \mathcal{L}_\phi^p \) to \( \mathcal{L}_\psi^q \) for \( 1 < p < q < \infty \). Related results were proved earlier by E. Nakai [8]. Some of the results presented here is joint with Eridani and E. Nakai, and has been published recently in [4].

1. Introduction

For \( 0 < \alpha < n \), the (classical) fractional integral operator or the Riesz potential \( I_\alpha \), defined by

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy,
\]

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is known to be bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ provided that $1/p - 1/q = \alpha/n$, $1 < p < q < \infty$. The associated inequality

$$\|I_\alpha f\|_{L^q} \leq C_{p,q}\|f\|_{L^p}$$

is known as the Hardy-Littlewood-Sobolev inequality (see [12], p. 354).

Later, in [1, 10], it is shown that $I_\alpha$ extends to a bounded operator from the Morrey space $\mathcal{E}^{p,\beta}(\mathbb{R}^n)$ to $\mathcal{E}^{q,\gamma}(\mathbb{R}^n)$ where $1/p - 1/q = \alpha/n, -n/p \leq \beta < \alpha$. The Morrey space $\mathcal{E}^{p,\beta}(\mathbb{R}^n)$ is defined to be the set of all locally integrable functions $f$ on $\mathbb{R}^n$ for which

$$\sup_B \frac{1}{r^\beta} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty,$$

where the supremum is taken over all balls $B = B(a, r)$ in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of $B$. The earlier result can be recovered from the latter by taking $\beta = -n/p$, because $\mathcal{E}^{p,\beta}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

A further extension of the above result is obtained by E. Nakai [7], who showed that $I_\alpha$ is bounded from the generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbb{R}^n)$ to $\mathcal{M}_{q,\psi}(\mathbb{R}^n)$ for $1/p - 1/q = \alpha/n, 1 < p < q < \infty$ and appropriate functions $\phi$ and $\psi$ with $\psi(r) = r^\alpha \phi(r)$. Here the generalized Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbb{R}^n)$ is defined by

$$\mathcal{M}_{p,\phi} := \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p,\phi}} < \infty \}$$

where

$$\|f\|_{\mathcal{M}_{p,\phi}} := \sup_B \phi(r) \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p}.$$  

Now the classical result can be recovered from Nakai’s by taking $\phi(r) = r^{-n/p}$.

In this note, we shall discuss the generalized fractional integral operator $T_\rho$, defined for a suitable function $\rho : (0, \infty) \to (0, \infty)$ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} \rho(|x - y|) \frac{f(y)}{|x - y|^n} dy,$$

whenever this integral makes sense. In particular, we are interested in the boundedness of $T_\rho$ from the generalized Morrey space $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\psi}$. Note that for $\rho(t) = t^\alpha, 0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the fractional integral operator mentioned earlier.
Next, we shall also present some results for a modified version $T_\rho$, denoted by $\tilde{T}_\rho$, which is defined by the formula

$$\tilde{T}_\rho f(x) := \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,$$

where $B_0$ is the unit ball around the origin and $\chi_{B_0}$ is the characteristic function of $B_0$. For $\rho(t) = t^\alpha$, the operator $\tilde{T}_\rho = \tilde{I}_\alpha$ is well-defined for $0 < \alpha < n+1$, and is known to be bounded from $L^p$ to BMO when $p > 1$ and $\alpha = n/p$, from $L^p$ to $\text{Lip}_\beta$ when $p > 1$ and $0 < \alpha - n/p = \beta < 1$, from BMO to $\text{Lip}_\alpha$ when $0 < \alpha < 1$, and from $\text{Lip}_\beta$ to $\text{Lip}_\gamma$ when $0 < \alpha + \beta = \gamma < 1$ (see [8] and further references therein). Our interest here will be the boundedness of $\tilde{T}_\rho$ from the generalized Campanato space $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{q,\psi}$. The space $\mathcal{L}_{p,\phi} = \mathcal{L}_{p,\phi}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{L}_{p,\phi}} := \sup_B \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{1/p} < \infty,$$

where $f_B$ denotes the average of $f$ over $B$, that is, $f_B := \frac{1}{|B|} \int_B f(y) dy$.

Note that for the space $\mathcal{M}_{p,\phi}$, the function $\phi(r)$ is usually required to be nonincreasing and $r^n \phi^p(r)$ to be nondecreasing, while for the space $\mathcal{L}_{p,\phi}$, it is $\frac{\phi(r)}{r}$ that is required to be nonincreasing.

The generalized fractional integral operator $T_\rho$ and its modified version $\tilde{T}_\rho$ were first studied by Nakai [8]. Some extensions of Nakai’s results were obtained by Eridani [2], Eridani and Gunawan [3], Gunawan [5], and Eridani, Gunawan, and Nakai [4]. Related results may also be found in Kurata et al. [6]. Results presented here are summarized from [4, 5].

Throughout this note, $C$, $C_1$, $C_p$ and $C_{p,q}$ will denote positive constants, which may vary from line to line.

2. Preliminaries

In the definition of $T_\rho$, we assume that the function $\rho$ satisfies the following conditions:

\begin{align*}
(2.1) & \quad \int_0^1 \frac{\rho(t)}{t} dt < \infty; \\
(2.2) & \quad \frac{1}{2} \leq \frac{\rho}{\rho(s)} \leq 2 \Rightarrow \frac{1}{C_1} \leq \frac{\rho(s)}{\rho(t)} \leq C_1.
\end{align*}
Meanwhile, in the definition of $\tilde{T}_\rho$, we assume that $\rho$ satisfies (2.1) and (2.2) and the following two additional conditions:

(2.3) $\int_r^\infty \frac{\rho(t)}{t^2} \, dt \leq C_2 \frac{\rho(r)}{r}$ for all $r > 0$;
(2.4) $\frac{1}{2} \leq \frac{r}{s} \leq 2 \Rightarrow |\rho(s) - \frac{\rho(s)}{s}\rho(s)| \leq C_3 |r - s| \frac{\rho(s)}{s}.$

For example, the function $\rho(t) = t^\alpha$, $0 < \alpha < n$, satisfies (2.1), (2.2) and (2.4). If $0 < \alpha < 1$, then $\rho(t) = t^\alpha$ also satisfies (2.3).

A function $\rho$ satisfying (2.2) is said to satisfy the doubling condition (with a doubling constant $C_1$). If $\rho$ satisfies the doubling condition, then one may observe that

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} \, dt \sim \rho(2^k r)$$

for every integer $k$ and $r > 0$. Further, it follows from the doubling condition that

$$\rho(r) \leq C \int_0^r \frac{\rho(t)}{t} \, dt,$$

for every $r > 0$. Next, if $\rho$ satisfies (2.1)-(2.4), then we have

$$\int_{\mathbb{R}^n} \left( \frac{\rho(|x_1 - y|)}{|x_1 - y|^n} - \frac{\rho(|x_2 - y|)}{|x_2 - y|^n} \right) \, dy = 0$$

for every choice of $x_1$ and $x_2$ (see [8]). For such a function $\rho$, we see that the operator $\tilde{T}_\rho$ maps a constant to a constant, and so $\tilde{T}_\rho$ is well-defined from one generalized Campanato space to another.

In the next section, we shall involve the so-called Hardy-Littlewood maximal operator $M$, which is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy.$$

A classical result for $M$ is that it is bounded on $L^p$ for $1 < p \leq \infty$ (see e.g. [12]). In [7], Nakai showed that if $\phi$ satisfies the doubling condition and

(2.5) $\int_r^\infty \frac{\phi(t)}{t} \, dt \leq C \phi(r)$ for all $r > 0$,

for some $1 < p < \infty$, then there exists $C_p > 0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\phi}} \leq C_p \|f\|_{\mathcal{M}_{p,\phi}},$$

that is, $M$ is bounded on $\mathcal{M}_{p,\phi}$.
In our approach, we shall also involve Young functions and Orlicz spaces. A function \( \Phi : [0, \infty] \to [0, \infty] \) is called a Young function if \( \Phi \) is convex, \( \lim_{r \to 0^+} \Phi(r) = \Phi(0) = 0 \) and \( \lim_{r \to \infty} \Phi(r) = \Phi(\infty) = \infty \). Note that a Young function is always nondecreasing. Given a Young function \( \Phi \), we define \( \Phi^{-1}(r) = \inf \{ s : \Phi(s) > r \} \) (with \( \inf \emptyset = \infty \)). If \( \Phi \) is continuous and bijective, then \( \Phi^{-1} \) is nothing but the usual inverse function.

If a Young function \( \Phi \) satisfies
\[
0 < \Phi(r) < \infty \text{ for } 0 < r < \infty,
\]
then \( \Phi \) is continuous and bijective from \([0, \infty)\) to itself. In this case, the inverse function \( \Phi^{-1} \) is increasing, continuous and concave, and hence satisfies the doubling condition.

For a Young function \( \Phi \), we define the Orlicz space \( L^\Phi = L^\Phi(\mathbb{R}^n) \) to be the set of all locally integrable function \( f \) on \( \mathbb{R}^n \) for which
\[
\int_{\mathbb{R}^n} \Phi\left( \frac{|f(x)|}{\epsilon} \right) dx < \infty
\]
for some \( \epsilon > 0 \). Here \( L^\Phi \) is equipped with the norm
\[
\|f\|_{L^\Phi} := \inf \left\{ \epsilon > 0 : \int_{\mathbb{R}^n} \Phi\left( \frac{|f(x)|}{\epsilon} \right) dx \leq 1 \right\}.
\]
Note that for \( \Phi(r) = r^p, 1 \leq p < \infty \), we have \( L^\Phi = L^p \). For further properties of Young functions and Orlicz spaces, see e.g. [11]. For their relevance with our subject, see [8, 9].

3. The boundedness of \( T_\rho \)

In [9], Nakai proved that \( T_\rho \) is bounded from \( M_{1,\phi} \) to \( M_{1,\psi} \), under appropriate conditions on \( \phi \) and \( \psi \), particularly the assumption that
\[
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C\psi(r), \quad \text{for all } r > 0.
\]
Later, Eridani [2] showed that \( T_\rho \) is bounded from \( M_{p,\phi} \) to \( M_{p,\psi} \) for \( 1 < p < \infty \), under similar assumptions on \( \rho, \phi \) and \( \psi \). Note, however, that we cannot recover the known results for \( I_\alpha \) from these results.
Recently, Eridani and Gunawan [3] proved that $I_\rho$ is bounded from $M_{p,\phi}$ to $M_{p,\phi^{p/q}}$ for $1 < p < q < \infty$, under some assumptions on $\rho$ and $\phi$. Precisely, they proved the following theorem.

**Theorem 3.1** [3]. Suppose that $\rho$ is surjective and satisfies the doubling condition. Suppose also that $\phi$ satisfies the doubling condition, (2.5), and

$$
\int_0^r \frac{\rho(t)}{t} dt + \rho(r)^{q/(q-p)} \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C_\rho(r), \quad \text{for all } r > 0.
$$

Then there exists $C_{p,q} > 0$ such that

$$
\|T_\rho f\|_{M_{q,\phi^{p/q}}} \leq C_{p,q} \|f\|_{M_{p,\phi}}
$$

that is, $T_\rho$ is bounded from $M_{p,\phi}$ to $M_{q,\phi^{p/q}}$, for $1 < p < q < \infty$.

Although Theorem 3.1 generalizes the result for $I_\alpha$, the assumptions on $\rho$ and $\phi$ seem to be different from those made by Nakai [9]. The following theorem serves as a link between Eridani and Gunawan’s and Nakai’s results.

**Theorem 3.2** [5]. Suppose that $\rho$ and $\phi$ satisfies the doubling condition. Suppose also that $\phi$ is surjective, satisfies (2.5) and

$$
\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C_\phi(r)^{p/q}, \quad \text{for all } r > 0.
$$

Then there exists $C_{p,q} > 0$ such that

$$
\|T_\rho f\|_{M_{q,\phi^{p/q}}} \leq C_{p,q} \|f\|_{M_{p,\phi}}
$$

that is, $T_\rho$ is bounded from $M_{p,\phi}$ to $M_{q,\phi^{p/q}}$, for $1 < p < q < \infty$.

**Proof** [5]. For every $x \in \mathbb{R}^n$ and $R > 0$, we write

$$
T_\rho f(x) = \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy + \int_{|x-y|\geq R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy = I_1(x) + I_2(x).
$$
For $I_1(x)$, we have

\[
|I_1(x)| \leq \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy
\]

\[
\leq \sum_{k=-\infty}^{1} \int_{2^{k}R \leq |x-y|<2^{k+1}R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy
\]

\[
\leq C \sum_{k=-\infty}^{1} \frac{\rho(2^k R)}{(2^k R)^n} \int_{|x-y|<2^{k+1}R} |f(y)| \, dy
\]

\[
\leq C Mf(x) \sum_{k=-\infty}^{1} \rho(2^k R)
\]

\[
\leq C Mf(x) \sum_{k=-\infty}^{1} \int_{2^{k}R}^{2^{k+1}R} \frac{\rho(t)}{t} \, dt
\]

\[
= C Mf(x) \int_{0}^{R} \frac{\rho(t)}{t} \, dt
\]

\[
\leq C Mf(x) \phi(R)^{(p-q)/q}.
\]

Meanwhile, for $I_2(x)$, we have

\[
|I_2(x)| \leq \int_{|x-y| \geq R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy
\]

\[
\leq \sum_{k=0}^{\infty} \int_{2^{k}R \leq |x-y|<2^{k+1}R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy
\]

\[
\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^n} \int_{|x-y|<2^{k+1}R} |f(y)| \, dy
\]

\[
\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^n} \left( \int_{|x-y|<2^{k+1}R} |f(y)|^p \, dy \right)^{1/p}
\]

\[
\leq C \|f\|_{M_{p,q}} \phi \sum_{k=0}^{\infty} \rho(2^{k+1}R) \phi(2^{k+1}R)
\]

\[
\leq C \|f\|_{M_{p,q}} \phi \sum_{k=0}^{\infty} \int_{2^{k}R}^{2^{k+1}R} \frac{\rho(t)\phi(t)}{t} \, dt
\]

\[
= C \|f\|_{M_{p,q}} \int_{R}^{\infty} \frac{\rho(t)\phi(t)}{t} \, dt
\]

\[
\leq C \|f\|_{M_{p,q}} \phi(R)^{p/q}.
\]
Summing the two estimates for $I_1$ and $I_2$, we get

$$|T_\rho f(x)| \leq C [Mf(x) \phi(R)^{(p-q)/q} + \|f\|_{M_{p,\phi}} \phi(R)^{p/q}].$$

Since $\phi$ is surjective, we can choose $R > 0$ such that $\phi(R) = Mf(x).\|f\|^{-1}_{M_{p,\phi}}$, assuming that $f$ is not identically 0 and that $Mf$ is finite everywhere. Hence, for every $x \in \mathbb{R}^n$, we have

$$|T_\rho f(x)|^q \leq C Mf(x)^p \|f\|^{q-p}_{M_{p,\phi}}.$$

From this and the boundedness of the maximal operator $M$ on $M_{p,\phi}$, we obtain the desired inequality. (QED)

The next theorem is another generalization of the known results for $I_\alpha$ (see [4] for its proof).

**Theorem 3.3** [4] Suppose that $\rho$ satisfies (2.1) and (2.2). Suppose further that $\frac{\rho(r)}{r^\alpha}$ and $r^{-n/p} \int_0^r \frac{\rho(t)}{t} dt$ are almost decreasing, $\int_r^\infty \frac{\rho(t)t^{-n/p}}{t} dt \leq C r^{-n/p} \int_0^r \frac{\rho(t)}{t} dt$, and there exist Young functions $\Phi_1$ satisfying (2.6) and $\Phi_2$ such that

$$r^{-n/p} \int_0^r \frac{\rho(t)}{t} dt \sim \Phi_1^{-1}(r^{-n}) \quad \text{and} \quad \Phi_1^{-1}(r^{-n}) \Phi_2^{-1}(r^{-n}) \sim r^{-n/q}$$

for $1 < p \leq q < \infty$. If $\phi$ satisfies the doubling condition and

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C \psi(r), \quad \text{for all } r > 0,$$

then $T_\rho$ is bounded from $M_{p,\phi}$ to $M_{q,\psi}$.

**Note.** A function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be almost decreasing if there exists a constant $C > 0$ such that $\theta(r) \geq C \theta(s)$ for $r \leq s$.

4. **The boundedness of $T_\rho$ on Campanato spaces**

We now turn to the modified fractional integral operator $\tilde{T}_\rho$. In [8, 9], Nakai proved that $\tilde{T}_\rho$ is bounded from $\mathcal{L}_{1,\phi}$ to $\mathcal{L}_{1,\psi}$ for appropriate functions $\phi$ and $\psi$. For $\phi(r) = r^\beta$ with $0 \leq \beta \leq 1$, the space $\mathcal{L}_{1,\phi}$ reduces to BMO (when $\beta = 0$) or Lip$_\beta$ (when $0 < \beta \leq 1$). In this case, Nakai’s result covers the BMO–Lip$_\alpha$ and Lip$_\beta$–Lip$_\gamma$ results for $\tilde{I}_\alpha$. For $\phi(r) = r^\beta$ with $-n/p \leq \beta < 0$, $1 < p < \infty$, we have Eridani’s result
[2] which covers the other results for \( \widetilde{I}_\alpha \). The following theorem is an extension of Eridani’s (see [4] for its proof).

**Theorem 4.1** [4] Suppose that \( \rho \) satisfies (2.1)–(2.4), and that \( \phi \) satisfies the doubling condition and \( \int_1^\infty \frac{\phi(t)}{t} \, dt < \infty \). If

\[
\int_r^\infty \frac{\phi(t)}{t} \, dt \int_0^r \frac{\rho(t)}{t} \, dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt \leq C\psi(r) \quad \text{for all } r > 0,
\]

then \( \widetilde{T}_\rho \) is bounded from \( L^p,\phi \) to \( L^p,\psi \) for \( 1 < p < \infty \).

The results for \( T_\rho \) indicate that the modified fractional integral operator \( \widetilde{T}_\rho \) must also be bounded from \( L^p,\phi \) to \( L^q,\psi \) for \( 1 < p \leq q < \infty \) and appropriate functions \( \phi \) and \( \psi \). Indeed, we have the following analog of Theorem 3.3 for \( \widetilde{T}_\rho \).

**Theorem 4.2** [4] Suppose that \( \rho \) satisfies (2.1) – (2.4). Suppose further that \( \rho(r) = r^{\alpha} l(r)^\beta \), where \( \alpha = n/p - n/q \), \( \beta > 0 \), and \( l(r) = -1/\log r \) for small \( r \) and \( l(r) = \log r \) for large \( r \), so that \( \rho \) satisfies the

5. **Concluding remarks**

Through our work we have been able to extend the known results for the classical fractional integral operator \( I_\alpha \) and its modified version \( \widetilde{I}_\alpha \) to the boundedness of \( T_\rho \) on Morrey spaces and that of \( \widetilde{T}_\rho \) on Campanato spaces. Our results not only cover the known results for \( I_\alpha \), but also enrich the class of functions of \( \rho, \phi \) and \( \psi \) for which the operator \( T_\rho \) is bounded from the Morrey space \( M_{p,\phi} \) to \( M_{q,\psi} \), and the operator \( \widetilde{T}_\rho \) is bounded from the Campanato space \( \mathcal{L}_{p,\phi} \) to \( \mathcal{L}_{q,\psi} \).

To give an example, let \( 1 < p < q < \infty \). Take \( \rho(r) = r^{\alpha} l(r)^\beta \), where \( \alpha = n/p - n/q \), \( \beta > 0 \), and \( l(r) = -1/\log r \) for small \( r \) and \( l(r) = \log r \) for large \( r \), so that \( \rho \) satisfies the
doubling condition. Then \( \int_0^r \frac{\omega(t)}{t} \, dt \sim \rho(r) \) (see [8]). Now take \( \phi(r) = r^{-n/p} l(r)^{\beta q/(p-q)} \). Then \( \phi(r)^{(p-q)/q} = \rho(r) \), and one may check that \( \rho \) and \( \phi \) satisfy the assumptions in Theorem 3.2. Hence the associated operator \( T_\rho \) is bounded from \( M_{p,\phi} \) to \( M_{q,\phi^{p/q}} \).

Further examples that support our results can be found in [4].

**REFERENCES**


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