Coxeter groups and regular polyhedra

Coxeter groups by definition admit a very simple set of generators and relations. The set of generators is called $S$; each generator has order 2; and there are additional relations

$$(st)^{m(s,t)} = 1 \quad (s \neq t \in S),$$

with $m(s,t)$ either an integer at least 2 or infinity. The relations can be encoded by a graph with vertex set $S$, in which $s$ and $t$ are joined by $m(s,t) - 2$ edges. The corresponding Coxeter group is written $W(S)$.

Coxeter gave a simple geometric realization of $W(S)$, a nice geometric criterion for $W(S)$ to be finite, and a complete list of all the graphs for which $W(S)$ is finite. (Most of them are Weyl groups of compact Lie groups.)

What is slightly less well known is the relationship of Coxeter’s classification to the classification of regular polyhedra in $\mathbb{R}^n$.

**Theorem.** Suppose $P$ is a regular polyhedron in $\mathbb{R}^n$. Then the isometry group of $P$ is generated by $n$ reflections

$$S = \{s_1, \ldots, s_n\},$$

and is a Coxeter group $W(S)$. This defines a bijection between similarity classes of regular polyhedra, and finite Coxeter groups having a single-line graph

$$s_1 \overset{m(1,2)-2}\rightarrow s_2 \overset{m(2,3)-2}\rightarrow s_3 \cdots \overset{m(n-1,n)-2}\rightarrow s_n$$

where all $m(i,i+1)$ are at least 3.

This bijection displays a lot of geometric information very clearly. For example, each $k$-dimensional face of $P$ is the regular polyhedron corresponding to the first $k$ vertices $\{s_1, \ldots, s_k\}$. The “Schläfi symbol” of $P$ is $\{m(1,2), m(2,3), \ldots, m(n-1,n)\}$.

The fact that the list of finite Coxeter groups includes complicated exceptions only for small graphs corresponds to the fact that the regular polyhedra in dimension at least five are very simple.

I’ll describe Coxeter’s proof of this theorem.