NATIONAL UNIVERSITY OF SINGAPORE  
DEPARTMENT OF MATHEMATICS  
Ph.D. QUALIFYING EXAMINATION  
ALGEBRA (SAMPLE PAPER)  

Time allowed : 3 hours

Answer All Questions

1. [20 marks] Let $\varphi : M' \to M$ be a homomorphism of abelian groups.

(a) Suppose that $\alpha : L \to M'$ is a homomorphism of abelian groups such that $\varphi \circ \alpha = 0$ (an example is the inclusion $\mu : \text{Ker} \varphi \hookrightarrow M'$). Prove or disprove each of the following.

(i) There is a unique homomorphism $\alpha_0 : \text{Ker} \varphi \to L$ such that $\mu = \alpha \circ \alpha_0$.

(ii) There is a unique homomorphism $\alpha_1 : L \to \text{Ker} \varphi$ such that $\alpha = \mu \circ \alpha_1$.

(b) Show that there is a four-term exact sequence

$$0 \to \text{Ker} \varphi \xrightarrow{\mu} M' \xrightarrow{\varphi} M \xrightarrow{\epsilon} \text{Coker} \varphi \to 0.$$  

What is Coker $\varphi$?

(c) Dualize (a); that is, find and prove the corresponding statement about Coker $\varphi$.

2. [15 marks] Let

$$0 \to M' \xrightarrow{\mu} M \xrightarrow{\epsilon} M'' \to 0$$

be a short exact sequence of R-modules, with $M'$ finitely generated. Show that $M$ is finitely generated if and only if $M''$ is. Give an example with $M$ finitely generated but $M'$ not finitely generated.

3. [15 marks] Show that an $A$-module $P$ is projective if and only if for any short exact sequence

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

of $A$-module homomorphisms, the corresponding sequence

$$0 \to \text{Hom}_A(P, N') \xrightarrow{\text{Hom}_A(P, f)} \text{Hom}_A(P, N) \xrightarrow{\text{Hom}_A(P, g)} \text{Hom}_A(P, N'') \to 0$$

is exact.
4. [15 marks] Let \( V \) be a finite dimensional \( k \)-vector space, of dimension \( n \), and let \( \varphi : V \to V \) be an invertible \( k \)-linear transformation from \( V \) to itself. Let \( \mathcal{B} = (e_1, \ldots, e_n) \) be a basis of \( V \).

(a) Show that 
\[
\varphi(\mathcal{B}) := (\varphi(e_1), \ldots, \varphi(e_n))
\]
is also a basis of \( V \).

(b) Let \( A_{\varphi} \in M_n(k) \) be the matrix of \( \varphi \) with respect to \( \mathcal{B} \) and \( \mathcal{B} \). In terms of \( A_{\varphi} \), compute

(i) the matrix of \( \varphi \) with respect to \( \mathcal{B} \) and \( \varphi(\mathcal{B}) \);
(ii) the matrix of \( \varphi \) with respect to \( \varphi(\mathcal{B}) \) to \( \varphi(\mathcal{B}) \);
(iii) the matrix of \( \varphi \) with respect to \( \varphi(\mathcal{B}) \) to \( \mathcal{B} \).

Justify your answers with proofs.

5. [15 marks] For each of the following statements about field extensions \( F \subseteq K \subseteq L \), either prove or give a counterexample.

(i) If \( L \) is a splitting field for a polynomial over \( F \), then \( L \) is a splitting field for a polynomial over \( K \).
(ii) If \( L \) is a splitting field for a polynomial over \( K \), and \( K \) is a splitting field for a polynomial over \( F \), then \( L \) is a splitting field for a polynomial over \( F \).
(iii) If \( L \) is a splitting field for a polynomial over \( F \), then \( K \) is a splitting field for a polynomial over \( F \).

6. [10 marks] Find \( \text{Gal}(F(\zeta)/F) \) where \( \zeta = \cos(2\pi/n) + isin(2\pi/n) \), \( F = \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \), and \( n = 4, 6, 12 \) or a prime \( p \).

7. [10 marks] Let \( \mathcal{S} \) denote the category whose objects are sets and whose morphisms are the functions from sets to sets. Let \( P : \mathcal{S} \to \mathcal{S} \) be the functor that assigns to each set \( X \) its power set \( P(X) \) (i.e. the set of all subsets of \( X \)), and assigns to each morphism \( f : A \to B \) a corresponding morphism \( P(f) : P(B) \to P(A) \) given by
\[
P(f)(X) = f^{-1}(X) \quad \text{for each } X \subseteq B.
\]

Show that \( P \) is a representable contravariant functor.