1. Let $A$ be an $n \times n$ matrix and let $f(x)$ be a polynomial such that $f(A) = 0$. Prove that $f(x)$ is a multiple of the minimal polynomial of $A$.

2. Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{C}$. Prove that if $AB = BA$, then there exists a basis $B$ such that both $[A]_B$ and $[B]_B$ are upper triangular.

3. Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{C}$. Suppose that $A$ and $B$ are similar to each other over $\mathbb{C}$. Determine whether $A$ and $B$ are similar to each other in $\mathbb{R}$. Justify your answer.

4. Let $A$ be a $2 \times 2$ matrix over $\mathbb{Z}$ with characteristic polynomial $x^2 + 1$. Determine whether $A$ is similar to \[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\] over $\mathbb{Z}$. Justify your answer.

5. Find the Jordan canonical forms of all $9 \times 9$ matrices over $\mathbb{C}$ with minimal polynomial $x^2(x - 3)^3$.

6. Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. Prove that $A$ is diagonalizable if and only if the minimal polynomial of $A$ is a product of distinct linear factors.

7. Let $A$ and $B$ be two $n \times n$ matrices. Prove that $AB$ and $BA$ have the same eigenvalues.

8. Find the Jordan canonical form of \[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}.
\]
9. Let $V$ be a finite-dimensional inner product space and let $W$ be a subspace of $V$. Prove that $(W^\perp)^\perp = W$.

10. Let $V$ be a finite-dimensional inner product space over $\mathbb{C}$ and let $G$ be the set of all linear transformations $T$ such that

$$(Tv_1|Tv_2) = (v_1|v_2).$$

Prove that $G$ forms a group.

11. Let $T$, $A$ and $B$ be linear transformations of a finite-dimensional inner product space $V$. Suppose that

$$(Tv|w) = (v|Aw) = (v|Bw)$$

for all $v, w \in V$. Prove that $A = B$.

12. Consider $\mathbb{R}^2$ with the standard inner product space. Determine the set of all linear transformations $T$ such that

$$(Tv|Tw) = (v|w)$$

for all $v, w \in \mathbb{R}^2$. Justify your answer.

13. Consider $\mathbb{R}^4$ with the standard inner product space. Determine the orthogonal complement of $\{(1,1,1,1), (1,0,0,1)\}$.

14. Prove that every finite-dimensional inner product space has an orthonormal basis.

15. Let $A$ be an $n \times n$ matrix over $\mathbb{R}$. Prove that $A$ is a sum of two nonsingular matrices.

16. Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{C}$. Prove that $I - AB$ and $I - BA$ have the same determinant.

17. Let $A$ and $B$ be two $n \times n$ matrices over $\mathbb{C}$. Suppose that $A$ and $B$ have the same minimal and characteristic polynomials. Determine whether $A$ and $B$ are similar over $\mathbb{C}$.

18. Let $A$ be a $3 \times 3$ matrix over $\mathbb{C}$. Prove that $A$ and the transpose of $A$ are similar over $\mathbb{C}$. 

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19. Let $A$ and $B$ be two $2 \times 2$ matrices over $\mathbb{Z}$. Suppose that $x^2 + x + 1$ is the characteristic polynomial for both $A$ and $B$. Determine whether $A$ and $B$ are similar to each other over $\mathbb{Z}$. Justify your answer.

20. Find two matrices $A$ and $B$ over $\mathbb{C}$ such that
   (i) $A$ and $B$ are similar to each other in $\mathbb{C}$,
   (ii) $A$ and $B$ are not similar to each other in $\mathbb{R}$.