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Research Interests:

- Variational PDE models and fast algorithms in image processing
- Data mining algorithms

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Total Variation Based Methods for Image Restoration

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Image restoration refers to the process of recovering an ideal image from its corrupted version which is the one we actually observe. The corruptions may be due to noise along the way of acquisition, out-of-focus of the imaging device, or packet loss during transmission of an acquired image from one location to another. There are several components in the restoration of corrupted images. First, we need to specify what an "image" is and what "corrupted" means, i.e., to specify an *image model* and a *degradation model*. Next, we need to design a *restoration model* which reconstructs an image satisfying the image model and reverses the degradation process. While a great deal of research efforts has been dedicated to the design of novel restoration models to solve restoration problems arisen from new applications, fundamental image restoration tasks such as denoising, deblurring and inpainting remain the center of focus in developing new paradigms for image restoration. Having said so, there isn't any major breakthrough in the recent decade in the modeling of the degradation processes of these fundamental tasks. Interestingly, it is some new image models which revolutionize the basic design of restoration models. In this article, we review a functional analytic point of view to images and show how it leads to some new variational and partial differential equation based restoration models which are more effective than previous methods in reconstructing ideal images. Possibly due to its explicit connection to traditional mathematical areas such as partial differential equations, functional analysis, fluid mechanics etc., the area of image restoration attracted researchers across a wide range of areas to join the taskforce. As a result, it is definitely not exaggerate to say that the recent growth of the area of image restoration is tremendously faster than it has been in the past few decades.

A Functional Analytic Image Model

Beyond the fact that an image is a matrix (or a stack of matrices in case of a multi-channel image) of real values or, in continuum description, a scalar function (or a vector function) defined on a rectangular domain, it is often more useful to prescribe a generative model for it. One prominent example is the view point that an image is a realization of a Markov Random Field — a popular approach in the 80's. This image model naturally leads to Bayesian image restoration methods. A feature of the Bayesian approach is that it allows us to systematically incorporate prior statistical knowledge about images into a restoration model. Besides the aforementioned statistical image model, we would like highlight a more recent functional analytic description of images — an image is a function of bounded variation. Mathematically, the space of all functions on a domain Ω with bounded variation, denoted by $BV(\Omega)$, is defined by

$$BV(\Omega) := \{u \in L_2(\Omega) : TV(u) < \infty\}$$

equipped with the norm $\|u\|_2 + TV(u)$. Here, $TV(u)$ is the *total variation* of u defined by

$$TV(u) := \sup_{\mathbf{v} \in C^1(\Omega \times \Omega),\ \|\mathbf{v}\|_\infty \leq 1} \left\{ \int_\Omega u \operatorname{div} \mathbf{v} dx \right\}$$

The most distinct feature of such a function space is that it admits discontinuous functions as its elements but at the same time certain regularity is required. The ability to allow discontinuities is very important in image processing since they correspond to edges in an image which are in turn important to human perception.

Variational Restoration Models

With the bounded variation (BV) image model in mind, the next question is how to build restoration models on top of it. Experience in variational calculus suggests that a natural way is to employ *variational models*. Variational models exhibit the solution of restoration problems as minimizers of appropriately chosen functionals. The minimization technique of choice for such models routinely involves the solution of nonlinear partial differential equations (PDEs) derived as necessary optimality conditions. In concert to the BV model, the chosen functionals often involve a term which measures the total variation of the image. Take image denoising as an example, the degradation model (also known as the forward model) maybe taken as

$$u_0 = u + \eta$$

where u_0 is an observed noisy image, u is an ideal noisy free image, and η is an additive noise such as Gaussian white noise. A celebrated variational model, known as the Rudin-Osher-Fatemi model (ROF model), is formulated as:

$$\min_{u \in BV(\Omega)} \int_\Omega (u - u_0)^2 dx + \lambda \int_\Omega |\nabla u| dx$$

Here, the second term denotes the total variation of u and $\lambda \geq 0$ is a regularization parameter specified by the user. This model is proposed by Rudin, Osher and Fatemi in 1992 [1]. The idea is to reconstruct an image u that is closed to the observed one u_0 but at the same time its total variation is small. As we mentioned before, the distinctive feature of such a total variation based model is that the denoised image is allowed to possess edges. Of course, the restored image may or may not contain edges, depending on the presence of edges in the noisy image. However, the story would have been totally different if another function space is used. For example, it can be proved that solutions to the following slightly different model (based on the $H^1(\Omega)$ space) must contain no edges:

$$\min_{u \in H^1(\Omega)} \int_\Omega (u - u_0)^2 dx + \lambda \int_\Omega |\nabla u|^2 dx$$

In particular, edges present in the observed image will be destroyed after the denoising process. Thus, the success of a restoration process highly depends on both a proper restoration model and its underlying image model. While the ROF model is a very successful edge-preserving denoising model, there are some caveats that one needs to bear in mind when using it, for instance, staircasing and contrast loss. Much recent research is directed towards the fine-tuning of the ROF model against these caveats. We refer the reader to [2] for a recent review of some latest developments.

Modeling Complex Image Restoration Problems

In many practical situations, a corrupted image may be involved in more than one degradation process.

This poses new challenges to the design of restoration models. A fundamental question is: How do the degradation processes interact? If the processes are totally uncorrelated, then we can restore a corrupted image effectively by tackling the degradation processes one at a time. However, if the processes are coupled, then we need a new model which can capture the interactions between different processes and can reverse all the coupled processes simultaneously. In [3], we studied the problem where the observed image is noisy, blurry and contains missing or occluded pixels. We showed that there is a strong coupling between blurring and occlusion, especially near the interface between observed regions and missing regions. Even though in the absence of noise, experiments demonstrated that restoration done by deblurring followed by inpainting (a restoration techniques for filling-in missing/occluded pixels) or inpainting followed by deblurring often leads to very unsatisfactory results (objects in the reconstructed image cannot be recognized by human eyes).

We proposed a joint model which is capable of denoising, blind deconvolution (deblurring without the knowledge of the blurring function) and inpainting simultaneously. The proposed model is variational and uses total variation minimization to control the regularity of the restored image. The degradation model is given by

$$u_0 = k * u + \eta \quad \text{on } \Omega_{\text{obs}}$$
$$u_0 \text{ missing on } \Omega_{\text{miss}} = \Omega \backslash \Omega_{\text{obs}}$$

Here, the observed image u_0 is the sum of the noise component η and the convolution of a blurring function k and an ideal image u. The values of u_0 are observed in the region Ω_{obs} whereas those in the region Ω_{miss} are missing. The proposed restoration model reads as follows:

$$\min_{u,k} \int_{\Omega_{\text{obs}}} (k * u - u_0)^2\, dx + \lambda_u \int_{\Omega_{\text{obs}} \cup \Omega_{\text{miss}}} |\nabla u|\, dx + \lambda_k \int_{\Omega_k} |\nabla k|\, dx$$

Due to the lack of knowledge of the blurring function, we attempt to reconstruct it as well. A term measuring the total variation of k is added to regularize the reconstructed k. We showed that this model leads to much better restored images than existing methods which treat the degradation processes separately. To obtain a solution to the above minimization problem, we solve the first order optimality condition derived from the first variation (Fréchet derivative):

$$2\chi_{\Omega_{\text{obs}}} k' * (k * u - u_0) - \lambda_u \nabla \cdot \frac{\nabla u}{|\nabla u|} = 0 \quad \text{on } \Omega_{\text{obs}} \cup \Omega_{\text{miss}}$$

$$2u' * (k * u - u_0) - \lambda_k \nabla \cdot \frac{\nabla k}{|\nabla k|} = 0 \quad \text{on } \Omega_k$$

Here, $\chi_{\Omega_{\text{obs}}}$ is the characteristic function on Ω_{obs}. While we have a computational procedure to obtain a solution in reasonable time, there is still a great room for further improvement.

Numerical Methods for Solving Variational Problems

In [4], we proposed a fast $O(n \log n)$ algorithm for minimizing the highly nonlinear ROF objective in the 1-dimensional case where n is th e size of the signal. The novelty lies on the important observation that the solution can be obtained by *marching the regularization parameter*. We showed that the ROF model possesses a nice semi-group property: solution at a parameter $\lambda = \lambda_1 + \lambda_2$ can be obtained by first solving the minimization problem with λ_1 followed by treating the solution as a noisy signal and solving the minimization problem again but with the parameter λ_2. By induction, the original problem can be decomposed into N problems: $\lambda = \lambda_1 + \lambda_2 + ... + \lambda_N$. We proved that there exists a sequence of critical parameters $\lambda = \hat{\lambda}_1 + \hat{\lambda}_2 + ... + \hat{\lambda}_N$ such that $N < n$, and that the solution at $\hat{\lambda}_1 + ... + \hat{\lambda}_{i+1}$ can be computed from the solution at $\hat{\lambda}_1 + ... + \hat{\lambda}_i$ using only $O(1)$ operations. This parameter marching idea gives a novel paradigm in solving total variation minimizing problems.

Conclusion

Total variation based methods provide powerful basic tools to solve a wide class of image restoration problems. Current research focuses on refining the basic models to handle more complex situations, for instance, images with extensive textures or special kind of noise such as salt-and-pepper noise. Other orthogonal directions include studying the decomposition of an image into components induced by a variation model, and obviously, designing fast computational methods.

References

[1] L. Rudin, S. Osher, and E. Fatemi. Nonlinear Total Variation Based Noise Removal Algorithms. *Physica D*, 60:259—268, 1992.

[2] T. Chan, S. Esedoglu, F. Park, and A. Yip. Recent Developments in Total Variation Image Restoration. In *The Handbook of Mathematical Models in Computer Vision*, pp.17—32, Springer, 2005.

[3] T. Chan, A. Yip, and F. Park. Simultaneous Image Inpainting and Blind Deconvolution. International Journal of Imaging Systems and Technology, 15(1):92—102, 2005.

[4] A. Yip and F. Park. Solution Dynamics, Causality, and Critical Behavior of the Regularization Parameter in Total Variation Denoising Problems. *UCLA CAM Report* #03-59, 2003.