Braids: New mathematical insights on an old topic

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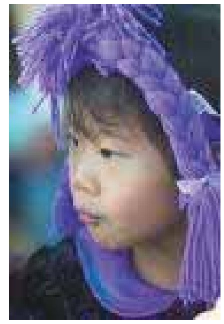
The notion of a braid as "anything plaited, interwoven, or entwined" goes back many centuries, and braids have been used more or less universally for decoration, art and fastening purposes, apparently since prehistory.



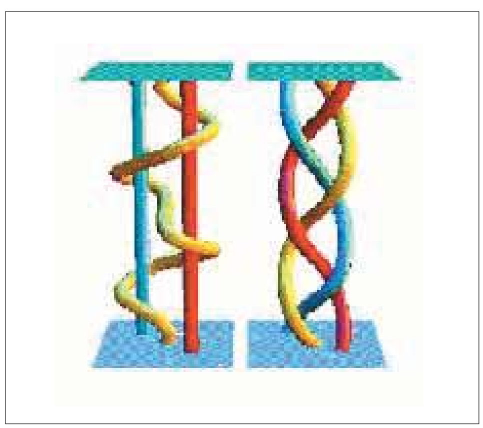
The oldest instruction book in English is over 500 years old; braiding as an art form (kumihimo) in Japan developed over 800 years before that.

Only over the past century or so have mathematicians tried to describe braids by means of abstract theory. Fortuitously, as the theory has developed, it has enabled applications to outstanding problems in physics, chemistry and biology.

Edo print of a seller of braids in a Kyoto shop.

The area of mathematics most suited to investigation of braids is called *topology*. Topology is the study of the shapes of geometric objects, which in applications may be as small as knotted DNA or long-chain polymers, or as large as the universe itself. Algebraic topologists attempt to distinguish such continuously varying objects by associating to them discrete, algebraic invariants; the process is comparable to capturing analogue data in digital format.

In topology, braid theory is an abstract geometric theory studying the everyday braid concept, and some generalizations. The idea is that braids can be organized into groups, in which the group operation is "do the first braid on a set of strings, and then follow it with a second on the twisted strings". Such *braid groups* may be described by explicit presentations, as was shown by E. Artin in 1925. For example, we may compare the two braids at right.



Two 3-strand brunnian braids.

To record these algebraically, we write *1* when the leftmost string passes in front of the middle strand, and $\underline{1}$ when it passes behind. Similarly, the notation is *2* or $\underline{2}$ according as the middle strand passes in front of or behind the rightmost strand. We list these crossings in order as we move

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After his first degree at Sydney University, Professor Berrick gained a DPhil from Oxford, where he became a fellow of St John's College. He then served as a lecturer at Imperial College London, before moving to NUS in 1981, and has been a professor here since 1996. His short-term research sojourns abroad have included Barcelona, Bielefeld, Cambridge, Kyoto, Lausanne, London, Oxford, Paris, Strasbourg, Sydney and Zurich. For the research described here, as well as other work in algebraic topology, Professor Berrick and his colleague Assoc. Prof. Wu Jie received the A*Star National Science Award in September 2007.

Research interests:

- Algebraic topology
- Algebraic *K*-theory
- Cohomology of groups

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down the braid from the top. Thus, the left braid is denoted

$$\underline{2211}2211,$$

while the right braid is

$$\underline{2}1\underline{2}1\underline{2}1.$$

Evidently, adjacent crossings *1* and $\underline{1}$ cancel each other, as do *2* and $\underline{2}$. Also, it's easy to see that the braid described as *121* may be moved into the position *212* simply by sliding, without any cutting or tying of strands, called an "equivalence". To decide whether such sliding can take us from the left braid to the right braid, we precede both by the braid $122\underline{1}2$, giving on the left

$$122\underline{1}2\underline{2211}2211,$$

and on the right

$$122\underline{1}2\underline{2}1\underline{2}1\underline{2}1.$$

By means of the algebraic cancellations and moves described above, we (or our computer) may reduce each of these notations to the "word" *211*. From Artin's theory, it now follows that the two pictured braids are equivalent. Thus, the theory enables a great deal of geometric data to be captured in a way that allows algorithmic discovery of key information. Further, the geometry suggests a number of algebraic questions that can be algorithmically challenging. Such challenges can be very useful in the design of codes that are difficult to break. Thus, the braid group is a natural object of study for cryptographers, in their quest to encode information, such as financial data on the internet, that can be decoded only by the "right" recipient.

The next breakthrough in the topological analysis of braids came with the realization, in the 1960s, that a braid with *n* strands can also be thought of as a path of a collection of *n* distinct particles moving through time, and which do not collide. Thus, the particles are collectively constrained to move within a space comprising *n* distinct points. Such a space, in which only certain configurations are permitted, is known as a *configuration space*.

The study of paths in configuration spaces has since become a very lively topic, well-suited to applications. For example, a familiar example of such a path is that of planes approaching an airport. In robotics, a machine is required not to bump into certain objects, and its permitted motion may depend on the nature of its flexible joints. Thereby, in medical applications, algorithms based on paths in configuration spaces can play a role in the planning of intricate surgery. Likewise, in computer animation, figures are seen as having more lifelike movement if they avoid certain "unnatural" configurations (like an elbow bent the wrong way). At the molecular level, structural rigidities constrain ligand-docking with proteins; which paths lead to a compound having the desired pharmacological properties? Similarly, protein-folding seems to be implicated in diseases like Alzheimer's and BSE; again, one needs to distinguish configuration space paths corresponding to preferred outcomes.

To return to the mathematics... Earlier this decade with two of my colleagues in the NUS Department of Mathematics, Wu Jie and Wong Yan Loi, together with Fred Cohen of University of Rochester, New York, we began a collaboration on braid groups. Our principal innovation was to address not just the *n*-stranded braid group in isolation, but to view the collection of all braid groups as a single entity, and study the effect on this object of removing strands. Centre stage in this approach came to be occupied by a kind of braid first noted by de Brun in 1895. A *brunnian braid* is one that becomes trivial (in effect sheds all crossings) when any of its strands is removed. Both braids of the pair pictured above are brunnian. One is reminded here of the borromean rings, a collection of three circles that cannot be slid apart, and yet, when any one of the three is removed, one finds that the two remaining circles are unlinked. There's a good reason for this association: by joining each top node to its counterpart underneath the braid, one obtains the borromean rings from the above brunnian braids.

From our examination of brunnian braids, which one can think of as occurring in familiar three-dimensional space, we obtained a quite unforeseen link with basic materials in higher-dimensional algebraic topology. These are the *homotopy groups of spheres*, groups that can be viewed as the building-blocks of that subject, because the geometric objects typically studied are built out of cells attached to each other by means of maps from one higher-dimensional sphere to another. (Such a sphere can be thought of as comprising the points in *n*-dimensional space that are equidistant from a given fixed point.) Anything that sheds light on a topic as important and difficult as the homotopy groups of spheres may be regarded as a breakthrough, especially when it involves something as relatively accessible as three-dimensional geometry.

Arising from this research, we recently held a program on braids at NUS' Institute for Mathematical Sciences:— http://www.ims.nus.edu.sg/Programs/braids/

The program featured a three-week summer school attended by more than forty graduate students and researchers, including some from outside mathematics, and an international conference with over 80 participants, among them most of the subject's leading experts. The meeting concerned both theory and applications, as may be seen from the titles below of some of its talks:

Braids and robotics

Braids, twist, writhe, and solar activity

Coloring *n*-string braids and tangles and its application to molecular biology

Length-based cryptanalysis of the braid group and some applications

References

Pictures above obtained from:-

http://www.ics.uci.edu/~eppstein/pix/boobash05/PurpleBraid.html

http://www.englisch.kumihimo.de/html/history.html

http://www.ucl.ac.uk/~ucahmab/home.html

Paper:-
A. J. Berrick, F. R. Cohen, Y. L. Wong & J. Wu: Configurations, braids, and homotopy groups, *J. Amer. Math. Soc.* **19** (2006), 265—326.