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**Research Interests:**

- Computational fluid mechanics
- Fluid structure interaction
- Speech processing

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# Numerical Methods for Incompressible Viscous Fluid and Fluid Structure Interaction

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Three years ago, mathematicians all over the world are celebrating the 300th anniversary of Leonhard Euler's birth. So the Euler equations that describe the motion of an ideal gas have been known for hundreds of years (255 years). The equations for viscous gas were laid down by Claude-Louis Navier 188 years ago. As fluids are so intimately related to our lives, over the past two hundred years, those equations have been studied by enormous people and have inspired many scientific streams.

However, in 1946, John von Neumann noticed [1] *"Our present analytical methods seem unsuitable for the solution of the important problems arising in connection with nonlinear partial differential equations .... The truth of this statement is particularly striking in the field of fluid dynamics.... The advance of analysis is, at this moment, stagnant along the entire front of nonlinear problems. That this problem is not of a transient nature but that we are up against an important conceptual difficulty .... yet no decisive progress has been made against them ...."*

Indeed, von Neumann had already made the decisive progress and found (founded) the new tool. That is the *computer*. He finished the *First Draft of a Report on the EDVAC* in the same year. The rest half century witnessed an everlasting flourish of numerical methods for fluid dynamics and other differential equations.

After so many years' development, some people (in particular engineers) think computational fluid dynamics is fully developed and the job has been completely taken over by engineering people. Why a mathematician nowadays should still be interested in developing new algorithms for such an old problem? Well, in the Moscow State University, the mathematics and the mechanics are in the same department. That indicates the strong and permanent connection between mathematics and mechanics. Indeed some existing algorithm takes time to digest and leaves rooms for further development. In my cases, mathematics gives qualitative estimation of the interaction between different quantities. That in return tells us how to treat different quantities economically but in a way that they deserve.

Here are the some examples. Consider the incompressible Navier-Stokes equations

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0.$$

We assume no-slip boundary condition  $\mathbf{u} = \mathbf{0}$  on the boundary  $\Gamma$ .  $\mathbf{u}$  is the velocity and  $p$  is the pressure. Constant  $\nu$  is the kinematic viscosity. The density is taken to be constant. The above equation describes the motion of an incompressible viscous Newtonian fluid like water or low speed gas. Many people consider

pressure as the Lagrange multiplier associated with the divergence free constraint. That leads to discretizations using inf-sup stable finite elements and one has to solve a large system that couples  $\mathbf{u}$  and  $p$  together. Further studies reveal that the pressure is a slave variable in the sense that (1) It is solely determined by velocity at the same instant [7]

$$\Delta p = \nabla \cdot \mathbf{g} - \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}), \quad \mathbf{n} \cdot \nabla p|_{\Gamma} = \mathbf{n} \cdot \mathbf{g} - \nu \mathbf{n} \cdot \nabla \times \nabla \times \mathbf{u}.$$

2) The pressure is strictly controlled by the viscosity term plus some lower order perturbation [2]. Once we know this relationship, we know that in a numerical algorithm pressure can be treated explicitly. The stability of the algorithm will not be jeopardized, yet we gain efficiency as we break a bigger system for  $(\mathbf{u}, p)$  together into two smaller systems for  $\mathbf{u}$  and  $p$  separately. Moreover, the above pressure equation is related to the commutator between Laplace and Leray projection operators. This new understanding leads to a systematic way of designing high order and stable schemes [3].

One thing that makes fluid mechanics fascinating is that related experiments are performed in front of your eyes at every single moment and every single day, and your questions never end. From my previous example, we know how to solve the system if velocity on the whole boundary is prescribed. But then every time when I open a faucet, I see the limitation of this condition. Because if I open several faucets at the same time, I have no way to prescribe a priori the outflux for each faucet. Mathematically, any pre-assumed mass-preserving outflux leads to a well-defined system. How can we single out the physical one that we have observed?

From elasticity, we know that a typical free boundary condition is traction-free. So, it is very natural to set the pseudo-traction of the fluid to equal to the ambient pressure along the outflow boundary:  $(\nu \nabla \mathbf{u} - p \mathbf{I}) \mathbf{n}|_{\Gamma_o} = -p_a \mathbf{n}$  where  $\Gamma_o$  is the outflow boundary and  $p_a$  is the ambient pressure. This turns out to be already known. What is new is now we can obtain a pressure Poisson equation with a Dirichlet type pressure boundary condition on  $\Gamma_o$ :

$$p|_{\Gamma_o} = \nu (\mathbf{n} \cdot \nabla \mathbf{u} - \nabla \cdot \mathbf{u}) + p_a$$

Once again we can show that the resulting pressure is controlled by the viscosity term. Hence first order semi-implicit scheme for the Navier-Stokes equations using the new pressure Poisson formulation is unconditionally stable [4]. The numerical result of flow in a bifurcated tube is shown in Figure 1.

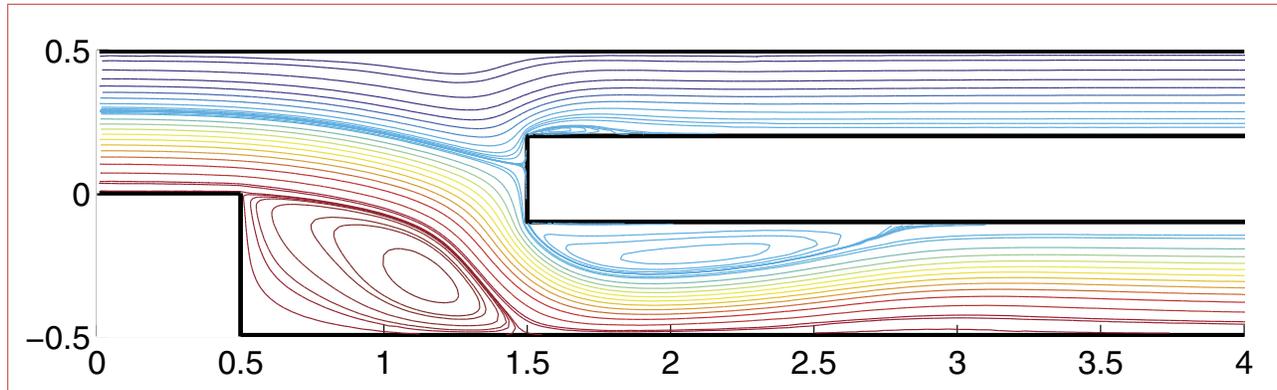


Figure 1: Streamline plot of flow in a bifurcated tube.

We do see phenomena related to fluid mechanics every day, but if we are more careful, most of them are related to the coupling between a fluid and its surrounding or enclosed solids (rigid or deformable). So, studying fluid structure interaction becomes natural. The first thing we should address is how to solve the fluid equations on a time varying domain. The idea comes from Truesdell [8] which I believe is a simplification or extension of the standard arbitrary Lagrangian Eulerian method. I have obtained a class of efficient and rather simple semi-implicit schemes for the time dependent Navier-Stokes equations in a time varying domain. Methods along this line are provable to be stable and accurate up to 5th order in time [5]. Now I am ready to address fluid structure interaction. Using a domain decomposition approach, the iteration that enforces the continuity of velocity and traction along the interface is proved to converge geometrically [6]. See Figures 2 and 3 for the benchmark tests of fluid structure interaction. ■

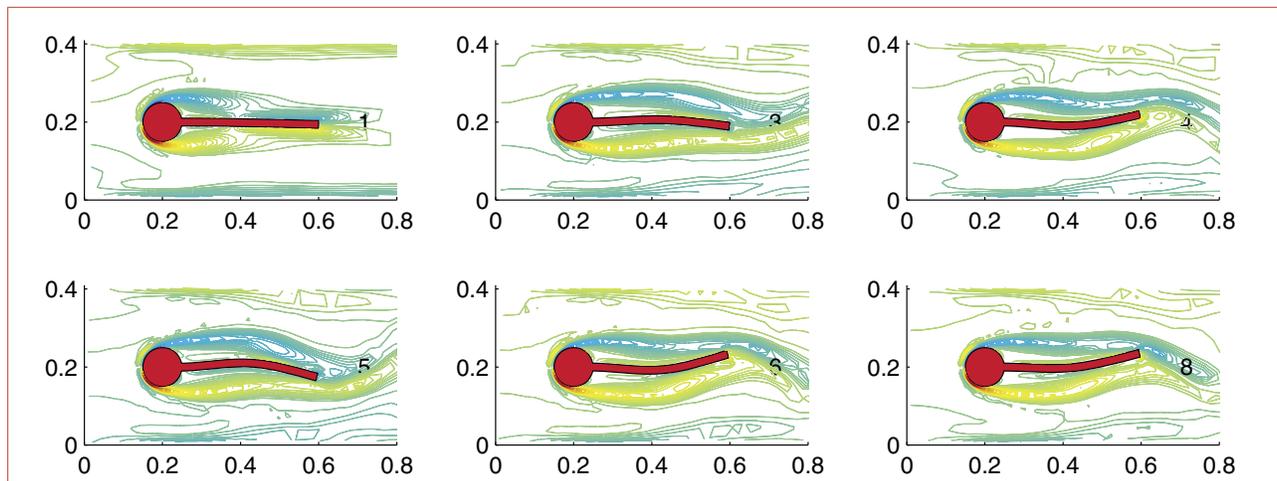


Figure 2: Vorticity contour plots of flow around an elastic bar attached to a rigid cylinder. Bar is of St. Venant-Kirchhoff type. Inflow from the left channel gradually starts before reaching a constant parabolic profile.

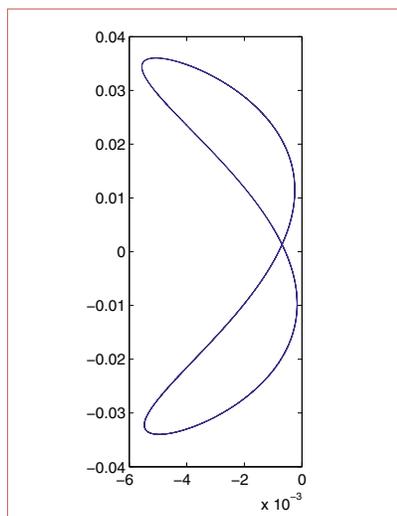


Figure 3: Final trajectory (displacement) of the middle point of the right bottom of the bar. It is not symmetric because there is gravity and the position of the cylinder slightly deviates from the middle of the channel.

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