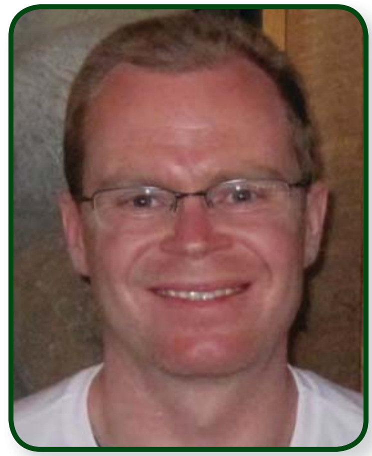
Towards Morse-Kirwan theory on singular spaces

Dr Graeme Wilkin
Department of Mathematics



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Morse Theory

Morse theory was originally developed in the 1920s by Marston Morse to study geodesics on a sphere. Since then it has undergone a number of improvements and generalisations in order to study different types of problems; for example Smale's proof of the Poincare conjecture in higher dimensions, Milnor's construction of exotic differentiable structures on spheres and Kirwan's method for computing the cohomology of symplectic and algebraic quotients.
A special case of the original problem studied by Morse is easy to visualise. Start with a standard two-dimensional sphere (for example, the surface of the earth). The shortest path between two points on the sphere runs along a segment of a great circle. These shortest paths are called geodesics. For example, two cities with the same longtitude will be connected by a great circle that runs north-south, and two cities on the equator will be connected by a geodesic that runs along the equator. Any other circle of constant latitude is not a geodesic. Viewed on a projection of the sphere to a flat plane (such as a map on your computer screen) this sometimes leads to surprising results: for example, the shortest path between Paris and Vancouver (both approx. 50 degrees north of the equator) passes over Iceland (64 degrees north).
Also of interest are the closed geodesics, i.e. those that form a closed loop on the sphere rather than a path between two distinct points. For example, the equator and circles of constant longtitude are closed geodesics.
Sometimes the notion of "shortest path" needs to be redefined to suit the application at hand. For example, an airplane would like to travel along the shortest path in order to save fuel, however factors such as prevailing winds or the recent volcanic eruption in Iceland may mean that it is more efficient to travel along a slightly different path. Mathematically, this change in perspective to looking for the most efficient path instead of the shortest path is expressed as a change in the metric, or distance measure on the sphere. One of the questions that Morse answered was: Given a smooth metric on the two-sphere, are there non-trivial closed geodesics?
The metric associates a real number (the length) to each loop on the sphere, and Morse's approach to this question was to consider the length function on the space of all loops on the sphere. The minimum corresponds to the set of trivial loops (a trivial loop is just a single point on the sphere), the non-minimal critical points correspond to the non-trivial closed geodesics, and Morse theory relates the topology of the total space (the space of all loops) to the topology of the critical sets. After developing the mathematical machinery to do this, Morse's theorem that nontrivial closed geodesics exist is then just the observation that the topology of the total space is different to the topology of the minimum (and therefore non-minimal critical points must exist).

Morse theory and the Yang-Mills equations

Physical systems naturally tend towards a minimum energy configuration, and Morse theory can sometimes provide a convenient tool for studying the topology of the space of minimum energy solutions. A famous example of this is contained in the work of Atiyah and Bott **[1]**, who used the Morse theory of the Yang-Mills functional to study the topology of the space of solutions to the Yang-Mills equations in two dimensions. The physical relevance of the Yang-Mills equations is in four dimensions; for example they play a key role in the Nobel prize-winning work of Salam, Glashow and Weinberg. In the two-dimensional case, rather than constructing a single solution and studying its properties, the goal is instead to understand the topology of the total space of solutions that lie at the minimum of the Yang-Mills functional (the ``energy'' function).

Academic Profiles:

1994-1999: B.Sc. (Hons) and B.Eng. (Hons), University of Melbourne
2001-2006: Ph.D in Mathematics, Brown University
2006-2009: J.J. Sylvester Assistant Professor, Johns Hopkins University
2009-2011: Burnett Meyer Postdoctoral Instructor, University of Colorado

Research Interest:

- Geometry and Topology
- Higgs bundles, quiver varieties and Yang-Mills theory

Contacts details
Telephone: 6516 7803
E-mail: matwgpd@nus.edu.sg
Office: S17 05-08

One would also like to extend Atiyah and Bott's theory to study related equations that are also physically relevant and of geometric interest. Two such equations are the two-dimensional Yang-Mills-Higgs equations and the equations defining a Nakajima quiver variety (which were introducted to parametrise Yang-Mills instantons over gravitational instantons). The Yang-Mills-Higgs equations were originally developed in the study of Higgs coupling, and the two-dimensional equations have found applications in mathematical fields such as geometric structures on surfaces, integrable systems and the proof of the fundamental lemma in the Geometric Langlands program. Topological information about Nakajima quiver varieties has been used to construct representations of quantum algebras **[2]**, which (among other applications) have been used in the study of the statistical mechanics of lattice models, such as the Ising model of a magnet **[3]**.

The obstruction to using Atiyah and Bott's methods to study quiver varieties and the Yang-Mills-Higgs equations is that now the total space is singular, and a priori no theory exists for the Morse theory of these energy functionals. Together with collaborators we have been able to overcome these difficulties to produce new results about the topology of the space of solutions to the Yang-Mills-Higgs equations in low rank **[4]**, **[5]**. The analogous program for quiver varieties leads to many interesting conjectures and questions, such as whether these techniques will shed new light on representations of quantum algebras and the mathematics of solvable lattice models.

References:

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