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From Coin Tosses to the Brownian Web and Net

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Introduction

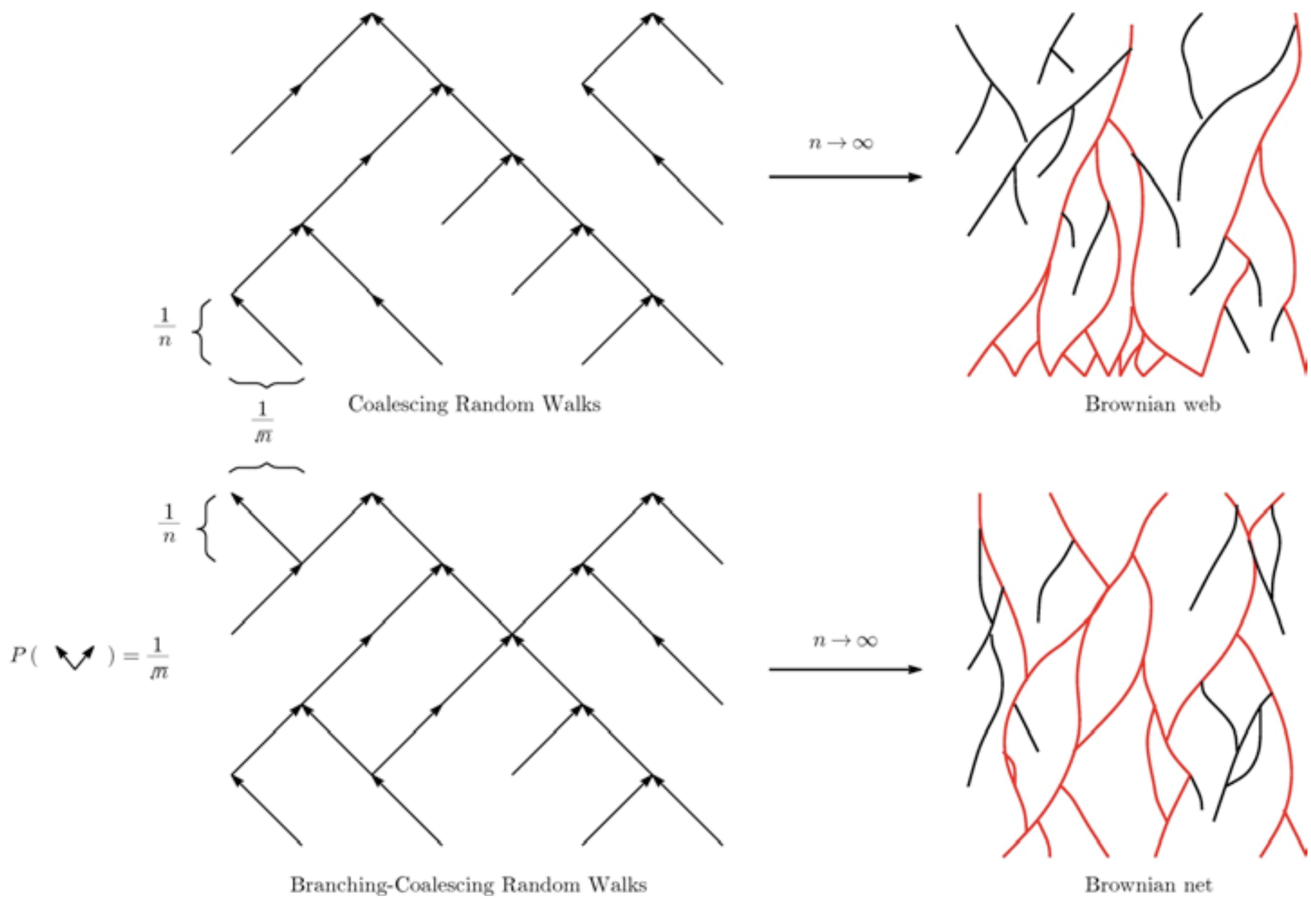
An important theme in probability theory is to identify universal phenomena on large space-time scales that are independent of the microscopic details. A classic example is the Central Limit Theorem. Take for instance the coin toss example, where a fair coin is tossed N times. If we plot the frequency at which k heads are observed against k, then we find that the frequency profile roughly follows a bell curve, where the center of the bell curve is at k=N/2, and the width of the curve is roughly the square root of N. The central limit theorem tells us that, as N tends to infinity, after centering and rescaling, the frequency profile converges to an exact bell-shaped function, which is the Gaussian density function. The power of the central limit theorem lies in the fact that the Gaussian density function is ubiquitous and not just restricted to the coin toss example. In a more abstract formulation, if X_1, X_2, … are the outcomes of a sequence of independent and identical experiments, then after proper centering and normalizing, the partial sum of these outcomes S_N=X_1+…+X_N will also asymptotically follow the Gaussian distribution. The Gaussian distribution is therefore universal in the sense that it governs, or more precisely, well-approximates the law of the fluctuation of the sum of a large collection of independent measurements, regardless of what one is actually measuring.

The central limit theorem can be extended to the functional level, which is called Donsker's invariance principle. Let us return to the coin toss example, and let X_1, X_2, … be the outcomes of the successive coin tosses, where X_i=1 if the i-th coin toss turns up head and X_i=-1 if it turns up tail. The partial sum S_N=X_1+…+X_N then measures the difference between the number of heads and tails. Instead of only observing S_N for a large N, let us observe the whole sequence (S_1, S_2, …, S_N) up to time N. The central limit theorem is concerned with the distribution of the last entry S_N. The functional central limit theorem tells us that if we speed up time by a factor of N and divide S_i by the square root of N, then the sequence (S_1, S_2, …, S_N), regarded as a function of time, asymptotically follows the distribution of a random function defined on the time interval [0,1], known as the Brownian motion. Originally introduced by biologists and physicists on an informal level to describe the random jiggling motion of pollen and dust particles in fluids and air, Brownian motion has since been rigorously constructed mathematically, and the functional central limit theorem establishes it as the universal random process governing the large space-time scale fluctuation of the sums of a large collection of independent measurements observed over time. The universality of Brownian motion makes it an ideal candidate to model many random motions in life and nature, ranging from the motion of molecules to stock prices.

To model complex phenomena involving multiple particles, individuals, or agents, we often need to employ more than one sequence of coin tosses, whose partial sum process S_N=X_1+…+X_N is also called a random walk, because we can think of a drunkard moving on the integer lattice Z, where every step he takes just equals the random coin toss X_i. The partial sum S_n records the position of the random walker at time n. A random walk can thus model the motion of an individual moving in space, and many interesting phenomena arise when there is a population of individuals interacting with each other, such as the spread of an infectious disease among a population. Various types of interactions can be introduced between the moving individuals modeled by random walks. For example, when two random walks meet, they can annihilate each other which can model the reaction of two chemical agents that become inert after reaction, or they can coalesce into a single random walk which can model the merging of two genealogical lines, or they can give birth to new random walks. Such interacting particle systems have been used to model many phenomena arising from physics, chemistry, biology etc, although their rigorous mathematical analysis is often difficult.

A special class of interacting random walks admits an analogue of the functional central limit theorem. They are the coalescing random walks on the integer lattice Z, where two walks merge into a single random walk whenever they meet. To construct the collection of coalescing random walks, for each point in the space-time lattice Z^2, we draw an arrow pointing either up-left or up-right with probability 1/2 each. In such an arrow configuration, space is plotted horizontally and time vertically. A random walk starting from a given lattice site simply follows the arrows upward in space-time, and because there is only a single arrow leading out of any site, two walks coalesce more Brownian motions starting from every point of the continuum space-time plane R^2. This random collection of coalescing Brownian motions is called the Brownian web. The construction and analysis of the Brownian web have been carried out by Arratia [1], Toth and Werner [5], Fontes, Isopi, Newman and Ravishankar [2]. Just like the Brownian motion, the Brownian web is also a universal object and arises as the scaling limit of general coalescing systems of particles in one dimension. The Brownian web has been used to model river networks, as well as the dynamics of domain walls of a one-dimensional ferromagnet at low temperature.

either an up-left or up-right arrow as in the construction of coalescing random walks, with probability one over square root of n, we draw both the up-left and up-right arrows, which represents a branching point. A random walk encountering one of these branching points splits into two random walks, with one following each of the two arrows. It turns out that if we rescale space by the square root of n and time by n as was done for coalescing random walks, then we obtain in the limit a random collection of branching-coalescing Brownian motions. We have named the limiting object the Brownian net due to the net-like structure appearing in it. The Brownian net is also



into a single walk whenever they meet. If we rescale space by a factor of square root of n and time by a factor of n, so that the lattice spacing tends to zero, then the functional limit theorem tells us that the random walk path starting from the origin will converge to a Brownian motion. However more is true. In fact the collection of all coalescing random walk paths, with one walker starting from every point of the space-time lattice Z^2, will converge to a limiting collection of coalescing Brownian motions, with one or

With coauthors Emmanuel Schertzer and Jan Swart [3, 4], we have been studying an extension of the Brownian web by allowing the extra effect of branching. Such branching effect may arise due to selection bias if the random walk paths model genealogical lines, or due to nucleation if the random walk paths model evolution of domain walls in magnets. A natural starting point is to consider branching-coalescing random walks on the space-time integer lattice Z^2. Instead of drawing

expected to arise as the universal scaling limit of general one-dimensional branching-coalescing particle systems.

The identification of universal large scale phenomena and the classification of these phenomena into different universality classes has been a central theme of modern probability theory, as well as statistical mechanics as epitomized by the renormalization group theory. The Brownian web and Brownian net are two instances of such universal limits, which extends the classical Brownian motion by incorporating many interacting Brownian motions. The verification that a particular discrete system converges to a universal limit is often not easy due to the many interacting components. However the notion of universality is powerful and attractive. Once a discrete system has been verified to belong to a given universality class, then it is known to share many essential properties with other models in the same universality class, and we can replace it either by the continuum limit or a different model in the same universality class which is more amenable to analysis.

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