The search for new Mathematical axioms

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Research Interests:

- Mathematical Logic
- Set Theory
- General Topology

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Introduction

Mathematicians prove theorems. But what does it mean to prove a theorem? The way it is commonly understood in Mathematics, it means to deduce a proposition expressed in a certain language from a set of axioms using a fixed collection of inference rules. This has been the ideal of Mathematics at least since the time of Euclid, though it was not always expressed in these terms. This formulation immediately raises the issue of how a proposition is recognized as an axiom. Traditionally axioms have been thought of as propositions that "neither need nor admit of proof". However, perceptions about which propositions need proof have changed through history. To take a famous example, Euclid considered his fifth postulate to be an axiom. In two dimensional geometry the fifth postulate says that given a line L and point p not on L, there exists exactly one line through p that does not intersect L. But mathematicians after Euclid thought that this needed proof, and many attempts were made to prove this from Euclid's other axioms. With the recognition of non-Euclidean geometries in the nineteenth century, it became clear that the fifth postulate was indeed independent of Euclid's other axioms; that is, it cannot be proved from his other axioms.

Poincaré Disk

It is instructive to examine in some detail how the existence of non-Euclidean geometries shows the independence of the fifth postulate. One of the simplest models of a geometry where all of Euclid's postulates except the fifth one hold is known as the Poincaré Disk. Our "space", the Poincaré Disk, consists of all points that lie in the interior of some fixed circle C – i.e. we omit the points that lie on the circle and take only the ones inside it. Given points P and Q in our space, the "line through P and Q" is interpreted as the arc of the circle through P and Q that intersects C at right angles (see Figure 1). With these interpretations of the words "point" and "line", it can be shown that the fifth postulate fails because given a line L and point p outside L, there will be more than one line passing through p that does not intersect L, while Euclid's other axioms all remain valid.

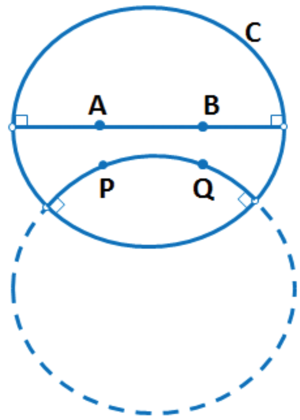

Figure 1: The Poincaré disk consists of the Points in the interior of the circle C. The "line through P and Q" and the "line through A and B" are shown. Note that the Points on the circle C represent infinity, so neither "line" has any endpoints.

At this point, you may object that when Euclid set down his axioms he had in mind a certain meaning for the words "point" and "line", and this is not what he meant. To be sure, Euclid did not have the Poincaré Disk in mind. However, the words "point" and "line" occur as primitive undefined terms in Euclid's axioms, and if all you have told me about points and lines are Euclid's first four axioms, then I am free to interpret these terms any way I want, as long as the four things you have told me about them come out true under my interpretation.

Or to put it another way, if all you know about points and lines are the first four axioms, then you do not know enough to be able to distinguish between the usual "flat" two dimensional space and the Poincaré Disk, and the existence of these two interpretations shows that the fifth postulate is independent of the other four because it holds in the usual interpretation while failing in the alternative one[1].

The above discussion raises an important point about the language in which mathematical propositions are expressed. This language has varied from time to time. For Euclid this language was Greek. No matter what this language is, it always contains some primitive undefined terms. The mathematician who uses these terms may have a certain picture of what they represent in his or her mind, but while proving a theorem only the information that is declared in the axioms may be used. The language of modern mathematics is an extremely precise and wholly artificial language in which only a small number of primitive terms occur. In fact, apart from some logical terms connected with reasoning in general, only one primitive term occurs: the notion of a set[2]. The rules for forming a meaningful sentence in this language are so precise, the axioms governing the usage of the primitive terms are so easy to list, and the rules of inference sufficiently well-defined, that it possible for a computer program to recognize when a certain string of symbols is a meaningful sentence in this language, and when a sequence of such sentences constitutes a proof. Needless to say, mathematicians do not think in terms of this artificial language, but all theorems and proofs can, at least in principle, be translated into it.

Notion of a Set

Why did modern mathematics evolve such an artificial language and what is the notion of a set? This is a complex story, but a very brief outline is as follows. In the nineteenth century several mathematicians felt a need for greater rigour in dealing with the concepts of calculus such as the concept of a limit, which till then were based on geometric intuition. This led them to redefine these notions in terms of just real numbers and operations performed on them. Then real numbers were reduced to the rational numbers, which in turn were reduced to the familiar natural numbers of arithmetic. This trend went hand in hand with two other trends. The first was an expansion in the sense of the term "function". Functions, which had initially denoted certain mathematical expressions, came to denote arbitrary correspondences. This led Frege to think of predicates and relations as functions that map objects to truth values. For example, he thought of the relation "is the father of" as a function whose arguments are a pair of objects x and y and it returns the value "true" if x is the father of y and the value "false" otherwise. Other functions could then be defined that took such functions as their arguments, and this process could be iterated. He built an artificial language using such functions and showed that this language was very useful for formalizing mathematical proofs. The second closely related trend was the study of arbitrary collections or sets of objects (usually mathematical objects such as real numbers), without any fundamental distinction being made between finite and infinite collections. This was initiated by Cantor, who showed that there were different kinds of infinite collections – some infinities were much bigger than others.

This trend towards studying arbitrary functions and sets eventually led to some paradoxes that resulted when certain sets with self-referential definitions were considered. To give a simple illustration of these paradoxes, suppose that there is a town in which there is a barber, who is male, and he shaves all and only those men in the town who do not shave themselves. Does the barber shave himself? By our hypothesis he shaves himself if and only if he does not. So such a barber simply cannot exist. Similarly it was realized that sets with such self-referential definitions cannot exist either, and axioms specifying exactly what kinds of sets exist were drawn up. Thus was born the axiom system known as ZFC, which is expressed in the artificial language referred to above. The axioms of ZFC provide all the information about sets that is relevant to mathematics, including information about what sets exist. All theorems proved by mathematicians today can be translated into the language of ZFC and proved from its axioms[3].

ZFC Axiom System

But is it the case that the axioms of ZFC can, at least in principle, be used to settle all mathematical questions? The answer is no, and in fact this answer is not unique to ZFC. In the 1930 Gödel proved that there there are propositions in the language of ZFC that are independent of the axioms of ZFC. That is, there are sentences in the language of ZFC that stand in the same relation to its axioms as Euclid's fifth postulate stands in relation to his other four – these sentences can neither be proved nor disproved using the axioms of ZFC. So for any one of these independent sentences, there exist two different interpretations of the term "set", one in which the sentence is true and another one in which the sentence is false, even though all the axioms of ZFC hold under both interpretations. All of the techniques of modern mathematics can be implemented with ZFC; therefore these independent sentences express mathematical questions that cannot be resolved using any mathematical technique, unless of course, one is willing to accept new axioms. It is worth noting that this predicament is not unique to ZFC or to the concept of "set". Gödel showed that for any axiom system that is powerful enough for the development of modern mathematics and whose axioms are simple to list, there will be sentences in the language of that axiom system that are independent of it, and this is regardless of the primitive concepts used[4].

A large part of modern set theory is devoted to constructing various different interpretations of the term "set". Just as the

1. For more on non-Euclidean geometries see [1]. You may also wish to google "Escher's Circle Limit ".
2. Technically, it is the relation of set membership.
3. Consult [2] for the original sources relating to the development of the artificial language in which ZFC is expressed, for the development of the set concept, and for the evolution of the axioms of ZFC.
4. Gödel's original papers are in [2].

interpretations of the terms "point" and "line" were carefully constructed in the Poincaré Disk example to ensure that Euclid's first four postulates were true, these interpretations of "set" are carefully constructed to ensure that the axioms of ZFC are all true. Such an interpretation is known as a model of ZFC. Now, given a mathematical sentence, if there is a model of ZFC in which that sentence is true and also a model where it is false, then the question of whether that sentence is true cannot be answered by usual mathematical techniques. One of the aims of investigating models of ZFC is this kind of delimitation of possibilities. It is worth noting here that these models are constructed using tools available in ZFC (this is similar to the Poincaré Disk being defined using Euclidean circles)[5]. So this is an exploration of the limits of mathematics from within itself. Another aim of this study is to discover new possible axioms. The traditional view of axioms was that they have to be self-evident. However, we may also discover sentences independent of ZFC, which when added as new axioms give such a coherent and appealing picture of some class of mathematical structures that we are tempted to enlarge ZFC in that direction. In fact, two distinct types of such sentences have been recognized over the last several decades of research. The first class may be called "minimality hypotheses". Very roughly, they say that the universe of sets is as thin as possible. The picture they present of certain infinite combinatorial structures is one of "non—structure", meaning that they imply that such combinatorial structures are disparate and impossible to classify. The other class of sentences are collectively known as "forcing axioms". Very roughly, they say that the universe of sets is as wide as possible. They tend to imply that a structure theory exists for several classes of combinatorial structures, meaning that the objects in these classes are not so diverse and must belong to one of a small number of types[6]. Another aim of studying models of ZFC is to discover new theorems of ZFC. Sometimes showing that a statement is independent of ZFC can suggest candidates for new theorems and can provide hints as to what techniques to use for proving them. In [4] Shelah calls this the "Rubble Removal Thesis": after the independent statements are cleared away, we may be left with unexpected theorems.

In my own work, I have made small contributions to this on-going study of models of ZFC. To illustrate, in [5] a limit on possibilities is set by showing that the existence of a certain set of real numbers is independent of ZFC. In [6], I have studied the consequences of some forcing axioms. In [7], I prove a theorem of ZFC, which could have been discovered classically, but was found only after several independence results in its vicinity were proved, and its proof uses techniques that are reminiscent of independence proofs. ■

5. There is a technical issue that is being swept under the rug here. Strictly speaking, it is not possible to construct a model of ZFC within ZFC, but only models of arbitrarily large finite fragments of it. This turns out to be good enough.

6. For more on forcing axioms and structure theorems for classes of combinatorial structures, see [3].

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