Howe Duality Conjecture

An Informal Discussion

by Gan Wee Teck

This article is concerned with a conjecture in representation theory which was formulated by Roger E. Howe [5] of Yale University in the mid-1970s. Roger Howe has had a definitive impact on the development of mathematics in Singapore in the last 25 years. Indeed, no fewer than three of his former PhD students are currently on the faculty of the National University of Singapore (NUS), including its Provost (Prof Tan Eng Chye), its Head of Department (HOD) of Mathematics (Prof Zhu Chengbo) and one of its deputy HODs (Prof Lee Soo Teck). In addition, the work of at least three other mathematicians at NUS are closely related to and inspired by that of Howe, including Associate Professor Loke Hung Yean, Associate Professor Martin Weissman (at Yale-NUS College) and the author of this article. Having such a closely knitted group of researchers has helped propel Singapore to prominence as a center of research in representation theory.

The subject area in which the Howe duality conjecture resides is called theta correspondence. As the name implies, this area has its origins in the theory of theta functions, which were discovered by Jacobi in the first half of the 19th century. In an influential paper [9] in 1964, Andre Weil recasted theta functions in the framework of representation theory (which is concerned with the action of groups as linear transformations on vector spaces). In particular, Weil interpreted theta functions as vectors in an infinite-dimensional representation Ω of a so-called metaplectic group *Mp(W)*, typically called the Weil representation or the oscillator representation. The terminology "oscillator" refers to the fact that the Weil representation occurs naturally in the quantum mechanical description of the simple harmonic oscillator.

Representation theory was created in the late 19th century by Frobenius and Schur. As mentioned above, it is about classifying all possible ways a group can act as linear transformations on vector spaces. It turns out that in many good situations, one can break up a given representation till one arrives at pieces which cannot be decomposed further, much as one breaks up a molecule into its constituent atoms. These so-called irreducible representations are thus the atoms of representation theory and are the fundamental objects to classify. The Weil representation Ω is an essentially irreducible representation of *Mp(W)*.

Representation theory provides the language and tools in the study of invariant theory, which concerns a study of multilinear tensors on finite-dimensional vector spaces which are invariant under a group action and was intensively pursued by David Hilbert and his school at the turn of the 20th century. This study culminated in Hermann Weyl's classic book "The Classical Groups: their Invariants and Representations" published in 1939.

This brings us to Howe's paper "Transcending classical invariant theory" [5], which was written in the mid-1970s but only published much later. In it, Howe went beyond Weyl in considering invariant theory for infinite dimensional representations of Lie groups. At that point in time, such infinite dimensional representations had become fashionable and important because of their connection with number theory, this connection being encapsulated in the so-called Langlands program. More precisely,

(i) Howe introduced the notion of dual pairs: these are subgroups of *Mp(W)* of the form $G \times H$ where G and H are mutual centralizers of each other. He gave a classification of all such possible dual pairs, and a concrete description of them. A prototypical example is obtained as follows: if U is a quadratic space and V a symplectic space, then $W = U \otimes V$ is naturally a symplectic space, and $O(U) \times Mp(V)$ is a dual pair in $Mp(W) = Mp(U \otimes V)$, where $O(V)$ denotes the orthogonal group of V.

(ii) Next, Howe considered the restriction of the Weil representation of *Mp(W)* to *G* x *H* over a local field. As a representation of *G x H*, Ω is no longer irreducible: it will break up into irreducible pieces, i.e. its constituent atoms. Roughly speaking, we may write this decomposition as:

$$\Omega = \otimes\ \pi \otimes \Theta(\pi)$$

where the sum runs over irreducible representations π of *G* and Θ(π) denotes a representation of *H*.

We can now formulate:

Howe Duality Conjecture

For any irreducible representation π of *G*, Θ(π) is either zero or irreducible.

This somewhat innocuous-looking statement is quite fundamental, as it predicts a close relation between the spectral theory of *G* and *H*. This relation is called the local theta correspondence.

We recall some progress towards this conjecture since it was first conceived:

(i) The first breakthrough towards the conjecture was provided by Howe himself, who proved in his paper [5] that the conjecture holds for dual pairs over the field R of real numbers and the field *C* of complex numbers. He also proved special cases of the conjecture for dual pairs over so-called *p*-adic fields.

(ii) Subsequently, extending Howe's method, Waldspurger [8] showed in 1990 that the conjecture holds for all *p*-adic fields when $p \neq 2$.

(iii) Kudla [6] showed that the conjecture holds for a subclass of representations (the supercuspidal ones).

Thus, one essentially knows the truth of the conjecture, but the missing case $p = 2$ is a nuisance in applications. Moreover, the existing proof of Waldspurger [9] is very complicated and long. While many experts believe that a simple proof of this basic conjecture exists, none has ever been found. It is therefore slightly surprising that, in June 2014, the author and Shuichiro Takeda [4] succeeded in giving a short and simple proof (10 pages) of the Howe duality conjecture for all *p*.

The Howe duality conjecture is not the end of the story; rather, it is the beginning! After one knows its truth, a series of questions naturally follows:

(a) Decide precisely when Θ(π) is nonzero.

(b) If Θ(π) is nonzero, what is it? In other words, describe Θ(π) in terms of π.

Of course, people have been investigating these questions since Howe's work, as the Howe duality conjecture was largely known or expected. Let me close with some recent sample results concerning these questions, especially by my colleagues

(1) In a recent paper [7] by Binyong Sun and Chengbo Zhu, a conjecture of Kudla and Rallis concerning question (a) was resolved: this is the so-called conservation relation.

(2) In a paper [1] and a preprint [2] by Atsushi Ichino and the author, question (b) was fully addressed for dual pairs of almost equal size.

(3) In a recent work, Hung Yean Loke and Jiajun Ma (a recent PhD student of C.B. Zhu) determined Θ(π) when π is an epipelagic supercuspidal representation.

The local theta correspondence has many interesting applications. As an example, it is the main tool used in the paper [3] to establish the local Langlands conjecture for GSp(4) and in [2] to establish the remaining cases of the so-called Gross-Prasad conjecture. There is also a global analog of the theta correspondence (i.e. over number fields) which is very useful for constructing automorphic representations, a basic class of objects studied in the Langlands program. The local results mentioned above thus provide the necessary local preliminaries for applications to number theory.

References

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