Group Theory

1. Let $G$ be a cyclic group of finite order and let $A$ be a subgroup of $G$. Prove that $G$ is cyclic.

2. Let $G$ be a cyclic group of order $n$. Prove that $G$ is isomorphic to $\mathbb{Z}_n$.


4. Let $p$ be a prime and let $G$ be a group of order $p^2$. Prove that $G$ is abelian.

5. Let $p$ be a prime and let $P$ be a group of order $p^n$. Prove that the order of $Z(P)$ is at least $p$.

6. Determine all the subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$. Justify your answer.

7. Let $G$ be a group of order $n$. Prove that $G$ is isomorphic to a subgroup of $S_n$.

8. Let $A$ be a subgroup of index $p$ of a finite group $G$, where $p$ is the smallest prime divisor of $|G|$. Prove that $A$ is a normal subgroup of $G$.

9. Let $D_n$ be the dihedral group of order $n$. Find the center and all the conjugacy classes of $D_{10}$.

10. Let $x$ and $y$ be two elements of order 2 of a group $G$. Prove that the subgroup $\langle x, y \rangle$ is either abelian or dihedral.

11. Let $\sigma$ be an even permutation. Give necessary and sufficient conditions of the cycle decomposition of $\sigma$ such that $\text{Cl}_{S_n}(\sigma) = \text{Cl}_{A_n}(\sigma)$. Justify your answer.

12. Let $p > 3$ be a prime. Suppose further that 3 is not a divisor of $p - 1$. Prove that a group of order $3p$ is cyclic.
Ring Theory

1. Let $R$ be a commutative ring with identity and let $M$ be an ideal of $R$. Prove that $M$ be a maximal ideal if and only if $R/M$ is a field.

2. Let $R$ be a finite integral domain with identity. Prove that $R$ is a field.

3. Let $R$ be a principal ideal domain with identity and let $I$ and $J$ be two distinct prime ideals of $R$. Prove that $1 \in <I, J>$.

4. Determine whether $\mathbb{Z}[x]$ is a principal ideal domain. Justify your answer.

5. Determine whether $\mathbb{Z}[\sqrt{-5}]$ is a unique factorization domain. Justify your answer.

6. Prove that the characteristic of an integral domain is either 0 or $p$, where $p$ is a prime.

7. Let $D$ be an integral domain and let $F$ be its quotient field. Prove that $f(x) \in D[x]$ is irreducible in $D[x]$ if and only if it is irreducible in $F[x]$.

8. Let $M$ be a finitely generated module of $\mathbb{Z}$. Prove that $M$ is isomorphic to a direct sum of cyclic $\mathbb{Z}$-modules.

9. Let $M$ be a simple module of a ring $R$ and let

$$\chi : M \to M$$

be a module homomorphism. Prove that if $\chi \neq 0$, then $\chi$ is an isomorphism.

10. Let $R$ be a unique factorization domain. Prove that $x \in R$ is irreducible if and only if $x$ is prime.

11. Let $R$ be a unique factorization domain. Prove that $R[x]$ is a unique factorization domain.

12. Construct a ring $R$ such that $\text{GCD}(x, y)$ does not exist for some $x, y \in R \setminus \{0\}$. Justify your answer.
Field Theory

1. Let $p$ be a prime. Prove that $\mathbb{Z}_p$ is a field. For each $n$, construct a field of $p^n$ elements.

2. Let $E$ and $F$ be two fields of order $p^n$. Prove that $E$ and $F$ are isomorphic to each other.

3. Let $F$ be a finite field and let $\tau \in F$. Prove that $\tau = a^2 + b^2$, for some $a, b \in F$.

4. Let $p$ and $q$ be two different primes. Determine whether there exists a field isomorphism from $\mathbb{Q}(\sqrt{p})$ to $\mathbb{Q}(\sqrt{q})$. Justify your answer.

5. Let $E/F$ be a field extension and let $\tau \in E$. Prove that $F[\tau] = F(\tau)$ if and only if $\tau$ is algebraic over $F$.

6. Let $F$ be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Prove that there exists a field extension $E/F$ such that $f(x)$ can be factorized into product of linear factors in $E$.

7. Let $E$ be the set $\{ \tau \in \mathbb{C} : \tau$ is algebraic over $\mathbb{Q} \}$. Prove that $E$ is countable.

8. Determine whether $\cos \pi/n$ is algebraic over $\mathbb{Q}$. Justify your answer.

9. Let $w_n = e^{2\pi i/n}$. Determine $[\mathbb{Q}(w_n) : \mathbb{Q}]$. Justify your answer.

10. Let $E/F$ be a field extension. Suppose that $[F(x) : F]$ is odd. Show that $[F(x) : F] = [F(x^2) : F]$.

11. Let $p$ be a prime and let $F$ be a finite field of $p^n$ elements. Suppose that $F^\times = < x >$. Let $\chi$ be a field automorphism of $F$. Prove that $\chi(x) = x^{p^r}$ for some $r$.

12. Determine the set of field automorphisms of $\mathbb{Q}(\sqrt{p})$, where $p$ is a prime. Justify your answer.