1. (a) Let $A \in \mathbb{R}^{n \times m}$. Assume that $Q \in \mathbb{R}^{m \times m}$ is orthogonal. Prove

$$\|AQ\|_F = \|A\|_F, \quad \|AQ\|_2 = \|A\|_2.$$ 

(b) Let

$$A = \begin{bmatrix} 187.68 & 0 \\ 187.68 & 187.68 \\ 187.68 & 187.68 \\ 0 & 187.68 \end{bmatrix}.$$

(i) Find the singular values and singular vectors of $A$.

(ii) Express $A$ as the SVD form $A = U\Sigma V^T$, where $U$ and $V$ are orthogonal matrices and $\Sigma$ is a “diagonal” matrix.

2. Given $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$. Assume that $A$ is symmetric. Prove that the Gauss-Seidel iterative method for solving the linear system of equation $Ax = b$

is convergent for any initial guess $x^{(0)}$.

3. Let $P_j$ denote the subspace of functions on the interval $[0,1]$ defined by polynomials having degree $\leq j$, and consider the function $f \in P_3$ defined by

$$f(x) = x^3 - 2x^2, \quad x \in [0,1].$$

(i) Compute the least squares approximation to $f$ from $P_1$.

(ii) Compute the function $h \in \{1, \cos \pi x, \sin \pi x\}$ that interpolates $f$ at the points \{0, 1/2, 1\}.

4. (a) Given the function $f(x)$ at the following values:

$$f(1.8) = 3.12014, \quad f(2.0) = 4.42569, \quad f(2.2) = 6.04241, \quad f(2.4) = 8.03014, \quad f(2.6) = 10.46675.$$ 

Approximate $\int_{1.8}^{2.6} f(x)dx$ using Composite Simpson rule.

(b) Romberg integration is used to approximate

$$\int_0^1 \frac{x^2}{1 + x^3}dx.$$ 

If $R_{11} = 0.250$ and $R_{22} = 0.2315$, what is $R_{21}$?